3.3

Function Makeover

Transformations and Symmetry of Polynomial Functions

LEARNING GOALS

In this lesson, you will:

- Dilate, reflect, and translate cubic and quartic functions.
- Understand that not all polynomial functions can be formed through transformations.
- Explore differences between even and odd functions, and even and odd degree functions.
- Use power functions to build cubic, quartic, and quintic functions.
- Explore the possible graphs of cubic, quartic, and quintic functions, and extend graphical properties to higher-degree functions.

KEY TERMS

- polynomial function
- quartic function
- quintic function

M. C. Escher is a well-known artist with a unique visual perspective. Many of his works display elusive connections, peculiar symmetry, and tessellations. Tessellations are symmetric designs with a repeated pattern.

You can find many images of Escher's work on the World Wide Web. Take a look and enjoy! Make sure to take a close look, because things may not be as straightforward as they seem.

PROBLEM 1 Refer to the Reference Points

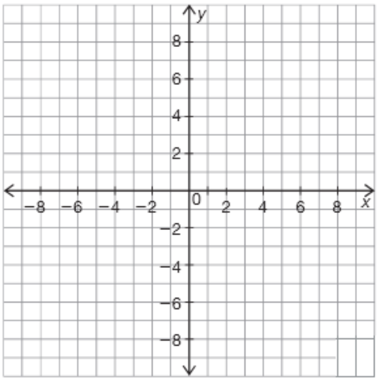
Recall that reference points are a set of points that are used to graph a basic function. Previously, you used reference points and the key characteristics of a parabola to graph the basic quadratic function. You learned that the reference points for the basic quadratic function are (0, 0), (1, 1), and (2, 4). The basic quadratic function is symmetric about the y-axis; that is, $f(x) = f(-x)$. Therefore, you can use symmetry to graph two other points of the basic function, (−1, 1), (−2, 4).

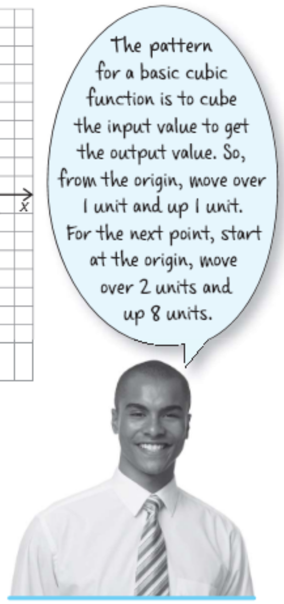
Let's consider a set of reference points and the property of symmetry to graph the basic cubic function.

To complete Questions 1 and 2, consider the basic cubic function, $f(x) = x^3$.

1. Complete the table for the given reference points. Then graph the points on the coordinate plane shown.

x	$f(x) = x^3$
0	
1	
2	





2. The graph of the basic cubic function is symmetric about the origin. So, $f(x) = -f(-x)$. Use the property of symmetry to determine 2 other points from the reference points. Then, use these points to graph the basic cubic function on the coordinate plane shown.

PROBLEM 2 Will Symmetry Prevail?

Transformations performed on a function $f(x)$ to form a new function $g(x)$ can be described by the transformational function:

$$g(x) = Af(B(x - C)) + D.$$

Previously, you graphed quadratic functions using this notation. You can use this notation to identify the transformations to perform on any function.

Recall that the constants A and D affect the *outside* of the function (the output values). For instance, if $A = 2$, then you can multiply each y-coordinate of $f(x)$ by 2 to determine the y-coordinates of $g(x)$.

The constants B and C affect the inside of the function (the input values). For instance, if $B = 2$, then you can multiply each x-coordinate of $f(x)$ by $\frac{1}{2}$ to determine the x-coordinates of $g(x)$.

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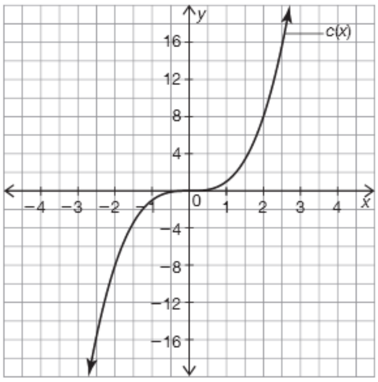
Function Form	Equation Information	Description of Transformation of Graph
$y = Af(x)$	$\|A\| > 1$	vertical stretch of the graph by a factor of A units
	$0 < \|A\| < 1$	vertical compression of the graph by a factor of A units
	$A < 0$	reflection across the x-axis
$y = f(Bx)$	$\|B\| > 1$	stretched horizontally by a factor of $\frac{1}{\|B\|}$
	$0 < \|B\| < 1$	compressed horizontally by a factor of $\frac{1}{\|B\|}$
	$B < 0$	reflection across the y-axis
$y = f(x - C)$	$C > 0$	horizontal shift right C units
	$C < 0$	horizontal shift left C units
$y = f(x) + D$	$D > 0$	vertical shift up D units
	$D < 0$	vertical shift down D units

1. Complete the table to show the coordinates of $g(x) = Af(B(x - C)) + D$ after each type of transformation performed on $f(x)$.

Type of Transformation Performed on $f(x)$	Coordinates of $f(x)$ → Coordinates of $g(x)$
Vertical Dilation by a Factor of A	$(x, y) \rightarrow$ (__________, __________)
Horizontal Dilation by a Factor of B	$(x, y) \rightarrow$ (__________, __________)
Horizontal Translation of C units	$(x, y) \rightarrow$ (__________, __________)
Vertical Translation of D units	$(x, y) \rightarrow$ (__________, __________)
All four transformations: A, B, C, and D	$(x, y) \rightarrow$ (__________, __________)

2. The graph of the basic cubic function $c(x) = x^3$ is shown.



a. Suppose that $g(x) = 2c(x)$. Use reference points and properties of symmetry to complete the table of values for $g(x)$. Then, graph and label $g(x)$ on the same coordinate plane as $c(x)$.

Reference Points on $c(x)$	→	Corresponding Points on $g(x)$
(0, 0)	→	
(1, 1)	→	
(2, 8)	→	

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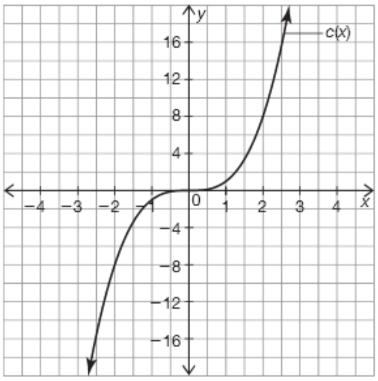
b. Suppose that $h(x) = \frac{1}{2}c(x)$. Use reference points and properties of symmetry to complete the table of values for $h(x)$. Then, graph and label $h(x)$ on the same coordinate plane as $c(x)$ and $g(x)$.

Reference Points on $c(x)$	→	Corresponding Points on $h(x)$
(0, 0)	→	
(1, 1)	→	
(2, 8)	→	

c. Describe the symmetry of $g(x)$ and $h(x)$. How does the symmetry of $g(x)$ and $h(x)$ compare to the symmetry of $c(x)$?

d. Determine whether $g(x)$ and $h(x)$ are even functions, odd functions, or neither. Verify your answer algebraically.

3. The graph of the basic cubic function $c(x) = x^3$ is shown.



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a. Suppose that $u(x) = c(2x)$. Use reference points and properties of symmetry to complete the table of values for $u(x)$. Then, graph and label $u(x)$ on the same coordinate plane as $c(x)$.

Reference Points on $c(x)$	→	Corresponding Points on $u(x)$
(0, 0)	→	
(1, 1)	→	
(2, 8)	→	

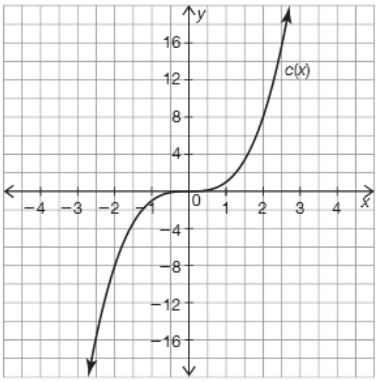
b. Suppose that $v(x) = c\left(\frac{1}{2}x\right)$. Use reference points and properties symmetry to complete the table of values for $v(x)$. Then, graph and label $v(x)$ on the same coordinate plane as $c(x)$ and $u(x)$.

Reference Points on $c(x)$	→	Corresponding Points on $v(x)$
(0, 0)	→	
(1, 1)	→	
(2, 8)	→	

c. Describe the symmetry of $u(x)$ and $v(x)$. How does the symmetry of $u(x)$ and $v(x)$ compare to the symmetry of $c(x)$?

d. Determine whether $u(x)$ and $v(x)$ are even functions, odd functions, or neither. Verify your answer algebraically.

4. The graph of the basic cubic function $c(x) = x^3$ is shown.



a. Suppose that $a(x) = -c(x)$. Use reference points and properties of symmetry to complete the table of values for $a(x)$. Then, graph and label $a(x)$ on the same coordinate plane as $c(x)$.

Reference Points on $c(x)$	→	Corresponding Points on $a(x)$
(0, 0)	→	
(1, 1)	→	
(2, 8)	→	

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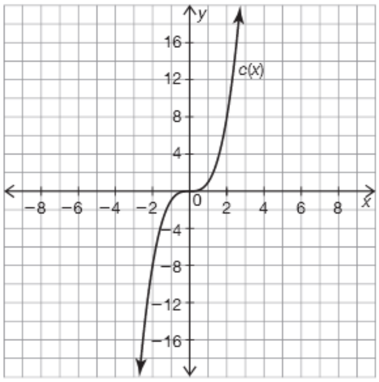
b. Suppose that $b(x) = c(-x)$. Use reference points and properties symmetry to complete the table of values for $b(x)$. Then, graph and label $b(x)$ on the same coordinate plane as $c(x)$ and $a(x)$.

Reference Points on $c(x)$	→	Corresponding Points on $b(x)$
(0, 0)	→	
(1, 1)	→	
(2, 8)	→	

c. Describe the symmetry of $a(x)$ and $b(x)$. How does the symmetry of $a(x)$ and $b(x)$ compare to the symmetry of $c(x)$?

d. Determine whether $a(x)$ and $b(x)$ are even functions, odd functions, or neither. Verify your answer algebraically.

5. The graph of the basic cubic function $c(x) = x^3$ is shown.



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a. Suppose that $m(x) = c(x - 5)$. Use reference points and properties of symmetry to complete the table of values for $m(x)$. Then, graph and label $m(x)$ on the same coordinate plane as $c(x)$.

Reference Points on $c(x)$	→	Corresponding Points on $m(x)$
(0, 0)	→	
(1, 1)	→	
(2, 8)	→	

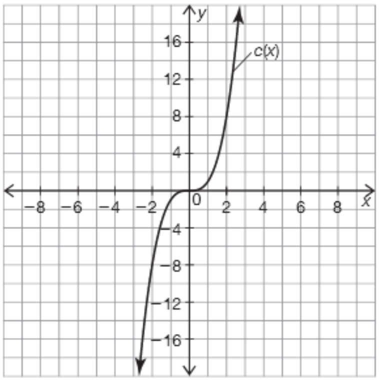
b. Suppose that $n(x) = c(x + 5)$. Use reference points and properties of symmetry to complete the table of values for $n(x)$. Then, graph and label $n(x)$ on the same coordinate plane as $c(x)$ and $m(x)$.

Reference Points on $c(x)$	→	Corresponding Points on $n(x)$
(0, 0)	→	
(1, 1)	→	
(2, 8)	→	

c. Describe the symmetry of $m(x)$ and $n(x)$. How does the symmetry of $a(x)$ and $b(x)$ compare to the symmetry of $c(x)$?

d. Determine whether $m(x)$ and $n(x)$ are even functions, odd functions, or neither. Verify your answer algebraically.

6. The graph of the basic cubic function $c(x) = x^3$ is shown.



a. Suppose that $j(x) = c(x) + 5$. Use reference points and properties of symmetry to complete the table of values for $j(x)$. Then, graph and label $j(x)$ on the same coordinate plane as $c(x)$.

Reference Points on $c(x)$	→	Corresponding Points on $j(x)$
(0, 0)	→	
(1, 1)	→	
(2, 8)	→	

b. Suppose that $k(x) = c(x) - 5$. Use reference points and properties of symmetry to complete the table of values for $k(x)$. Then, graph and label $k(x)$ on the same coordinate plane as $c(x)$ and $j(x)$.

Reference Points on $c(x)$	→	Corresponding Points on $k(x)$
(0, 0)	→	
(1, 1)	→	
(2, 8)	→	

c. Describe the symmetry of $j(x)$ and $k(x)$. How does the symmetry of $j(x)$ and $k(x)$ compare to the symmetry of $c(x)$?

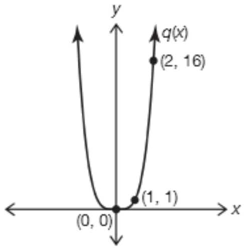
d. Determine whether $j(x)$ and $k(x)$ are even functions, odd functions, or neither. Verify your answer algebraically.

7. Complete the table to summarize the effects that transformations have on the basic cubic function $c(x) = x^3$. The first row has been completed for you.

Effects of Rigid Motions on the Basic Cubic Function $c(x) = x^3$			
Rigid Motion	**New Transformed Function $p(x)$ in Terms of $c(x)$**	**Description of Symmetry of $p(x)$**	**Is $p(x)$ Even, Odd, or Neither?**
Vertical Stretch Dilation	$p(x) = Ac(x)$, $A > 1$	Symmetric about the point (0, 0)	Odd
Vertical Compression Dilation			
Horizontal Stretch Dilation			
Horizontal Compression Dilation			
Reflection across x-axis			
Reflection across y-axis			
Vertical Translation			
Horizontal Translation			

8. Do you think that your results in Question 7 would be the same for *any* odd power function? Explain your reasoning.

9. The graph of the basic quartic function $q(x) = x^4$ and its reference points are shown. Use the graph to sketch the function after dilations, reflections, and translations. Pay special attention to the symmetry after the transformations. Record your conclusions by completing the table that follows. The first row has been completed for you.



Effects of Rigid Motions on the Basic Cubic Function $q(x) = x^4$			
Rigid Motion	**New Transformed Function $p(x)$ in Terms of $q(x)$**	**Description of Symmetry of $p(x)$**	**Is $p(x)$ Even, Odd, or Neither?**
Vertical Stretch Dilation	$p(x) = Ac(x)$, $A > 1$	Symmetric about the y-axis	Even
Vertical Compression Dilation			
Horizontal Stretch Dilation			
Horizontal Compression Dilation			
Reflection across x-axis			
Reflection across y-axis			
Vertical Translation			
Horizontal Translation			

10. Do you think that your results in Question 9 would be the same for *any* even power function? Explain your reasoning.

11. Use the appropriate word from the box to complete each statement.

always	sometimes	never

a. If a dilation is performed on an odd function $f(x)$ to produce $g(x)$, then $g(x)$ will

_______________ be an odd function.

b. If a reflection is performed on an even function $f(x)$ to produce $g(x)$, then $g(x)$ will

_______________ be an even function.

c. If a translation is performed on an odd function $f(x)$ to produce $g(x)$, then $g(x)$ will

_______________ be an odd function.

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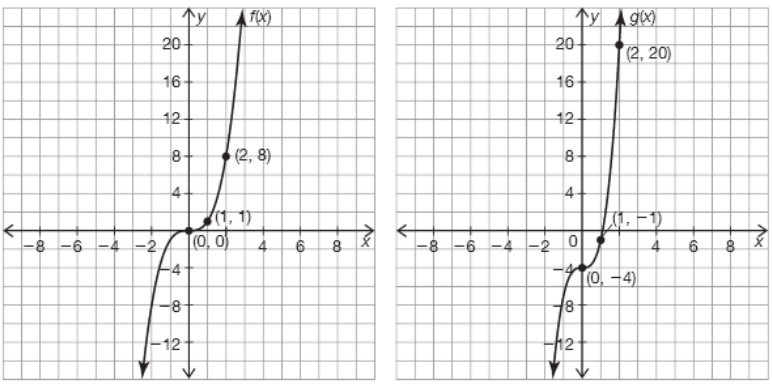
d. If a translation is performed on an even function $f(x)$ to produce $g(x)$, then $g(x)$ will

_______________ be an even function.

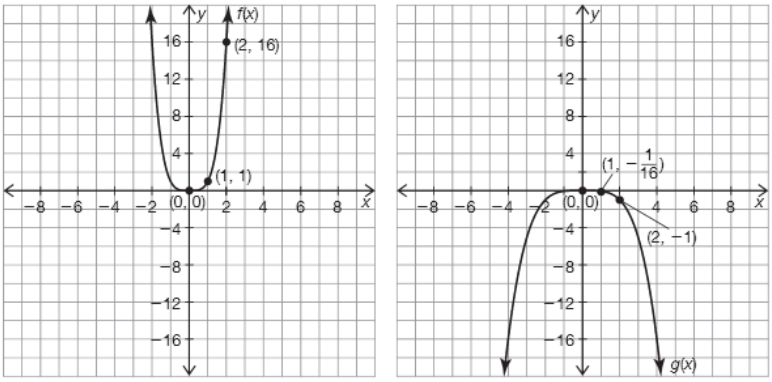
PROBLEM 3 Multiple Transformations

1. Analyze the graphs of $f(x)$ and $g(x)$. Describe the transformations performed on $f(x)$ to create $g(x)$. Then, write an equation for $g(x)$ in terms of $f(x)$. For each set of points shown on $f(x)$, the corresponding points after the rigid motions are shown on $g(x)$.

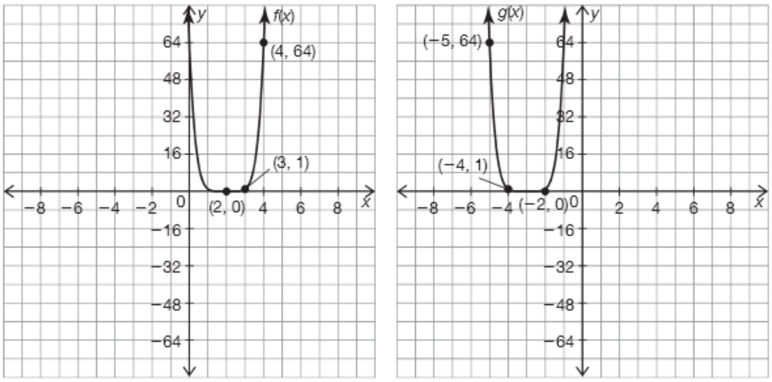
a. $g(x) =$ _______________



b. $g(x) =$ ____________



c. $g(x) =$ ______________



2. The equation for a polynomial function $p(x)$ is given. The equation for the transformed function $t(x)$ in terms of $p(x)$ is also given. Describe the transformation(s) performed on $p(x)$ that produced $t(x)$. Then, write an equation for $t(x)$ in terms of x.

a. $p(x) = x^5$

$t(x) = 0.5p(-x)$

b. $p(x) = x^4$

$t(x) = 2p(x + 3)$

c. $p(x) = x^3$

$t(x) = -p(x - 2) + 4$

PROBLEM 4 When Transformations Just Don't Cut It

A **polynomial function** is a function that can be written in the form

$$p(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_2 x^2 + a_1 x + a_0,$$

where the coefficients $a_n, a_{n-1}, \ldots a_2, a_1, a_0$ are complex numbers and the exponents are nonnegative integers. The form shown here is called the standard form of a polynomial.

You already know that a third-degree polynomial function has a special name—a cubic function. A **quartic function** is a fourth degree polynomial function, while a **quintic function** is a fifth degree polynomial function.

You can describe any linear or quadratic functions in terms of the transformations performed on the basic functions. Is this true for *any* polynomial function? That is, can you derive any polynomial function by transforming a basic function?

1. Consider the polynomial function $p(x) = x^3 + 2x^2 - 3x$.

   a. Predict what the graph of $p(x)$ looks like. Describe the number of x-intercepts and end behavior.

b. Use a graphing calculator to graph $p(x)$. Were your predictions accurate?

c. Can you describe which transformations were performed on $f(x) = x^3$ that results in the graph of $p(x)$?

Transformations on basic functions cannot be used to derive any polynomial. Therefore, you will need to consider another method.

Use each basic power function shown to complete Questions 2 through 6.

$f(x) = x$	$g(x) = x^2$	$h(x) = x^3$	$j(x) = x^4$	$k(x) = x^5$

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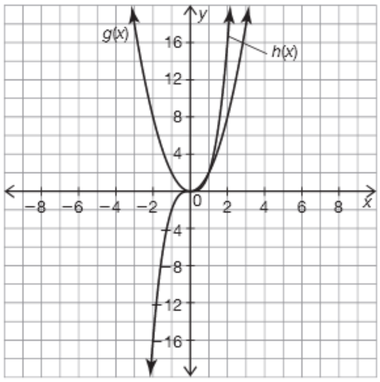
2. Consider the function $a(x)$, where $a(x) = h(x) + 2g(x)$.

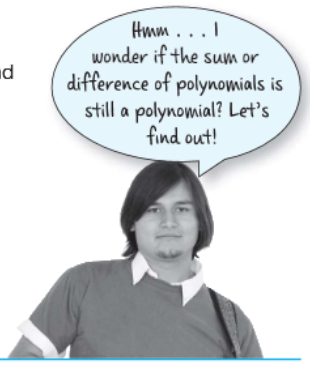
a. The functions $g(x)$ and $h(x)$ are shown. Complete the table of values and sketch $a(x)$ on the coordinate plane shown. The first row has been completed for you.



x	$h(x)$	$g(x)$	$a(x)$
−2	−8	4	$-8 + 2(4) = 0$
−1			
0			
1			
2			

b. Write the equation for $a(x)$.

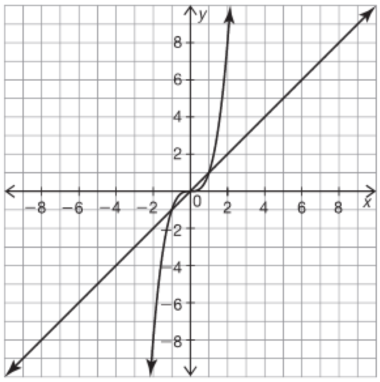
c. Explain any differences between the graph of $a(x)$ and the graph of the basic power function of the same degree as $a(x)$.



$f(x) = x$ $g(x) = x^2$ $h(x) = x^3$ $j(x) = x^4$ $k(x) = x^5$

3. Consider the function $b(x)$, where $b(x) = 2f(x) - h(x)$.

a. The functions $f(x)$ and $h(x)$ are shown. Complete the table of values and sketch $b(x)$ on the coordinate plane.



x	$f(x)$	$h(x)$	$b(x)$
−2			
−1			
0			
1			
2			

b. Write the equation for $b(x)$.

c. Explain any differences between the graph of $b(x)$ and the graph of the basic power function of the same degree as $b(x)$.

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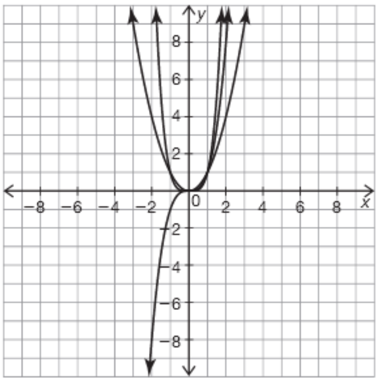
$f(x) = x$	$g(x) = x^2$	$h(x) = x^3$	$j(x) = x^4$	$k(x) = x^5$

4. Consider the function $c(x)$, where $c(x) = j(x) + 0.5h(x) - 2g(x)$.

a. The functions $g(x)$, $h(x)$, and $j(x)$ are shown. Complete the table of values and sketch $c(x)$ on the coordinate plane.



x	$j(x)$	$h(x)$	$g(x)$	$c(x)$
−2				
−1				
0				
1				
2				

b. Write the equation for $c(x)$.

c. Explain any differences between the graph of $c(x)$ and the graph of the basic power function of the same degree as $c(x)$.

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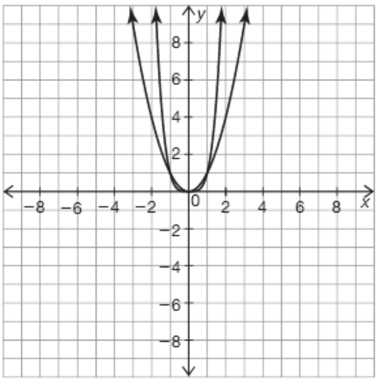
$f(x) = x$	$g(x) = x^2$	$h(x) = x^3$	$j(x) = x^4$	$k(x) = x^5$

5. Consider the function $d(x)$, where $d(x) = -j(x) + 3g(x) - 1$.

a. The functions $g(x)$ and $j(x)$ are shown. Complete the table of values and sketch $d(x)$ on the coordinate plane.



x	$j(x)$	$g(x)$	$d(x)$
−2			
−1			
0			
1			
2			

b. Write the equation for $d(x)$. Is $d(x)$ a polynomial?

c. Explain any differences between the graph of $d(x)$ and the graph of the basic power function of the same degree as $d(x)$.

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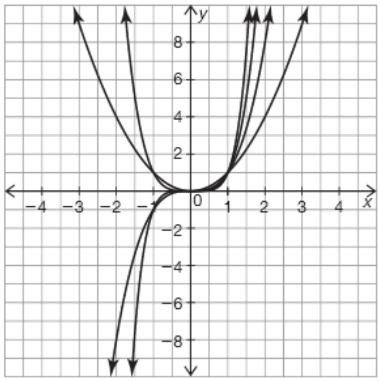
$f(x) = x$	$g(x) = x^2$	$h(x) = x^3$	$j(x) = x^4$	$k(x) = x^5$

6. Consider the function $z(x)$, where $z(x) = k(x) + 2j(x) - 4h(x) - 6g(x)$.

a. The functions $g(x)$, $h(x)$, $j(x)$, and $k(x)$ are shown. Complete the table of values and sketch $z(x)$ on the coordinate plane.



x	$k(x)$	$j(x)$	$h(x)$	$g(x)$	$z(x)$
−3					
−2					
−1					
0					
1					
2					
3					

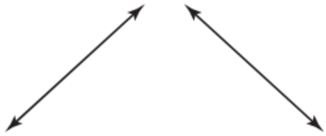
b. Write the equation for $z(x)$.

c. Explain any differences between the graph of $z(x)$ and the graph of the basic power function of the same degree as $z(x)$.

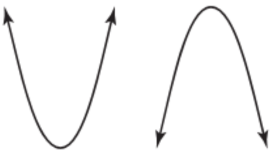
Talk the Talk

The possible shapes of linear, quadratic, cubic, quartic, and quintic functions are shown.

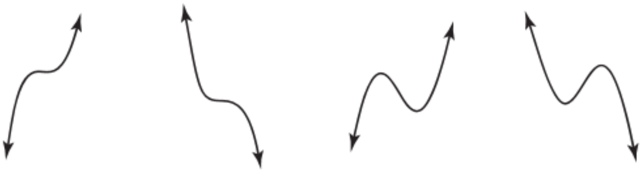
Linear Functions



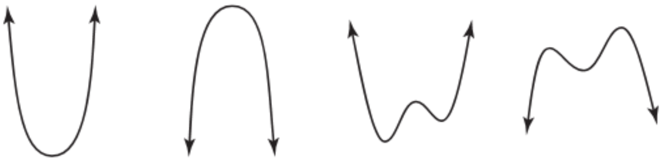
Quadratic Functions



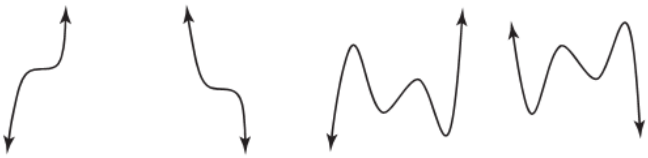
Cubic Functions



Quartic Functions



Quintic Functions



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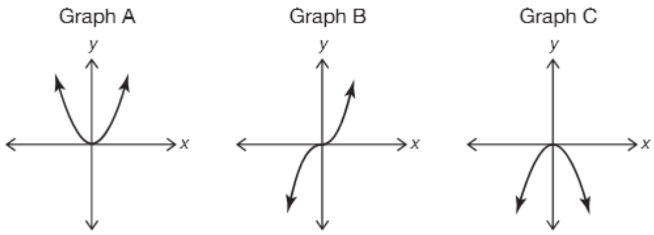
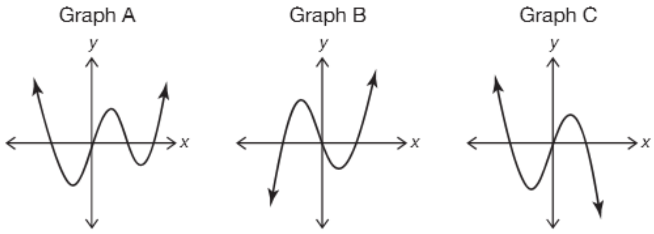
1. Choose the possible graph(s) for each given polynomial function $f(x)$.

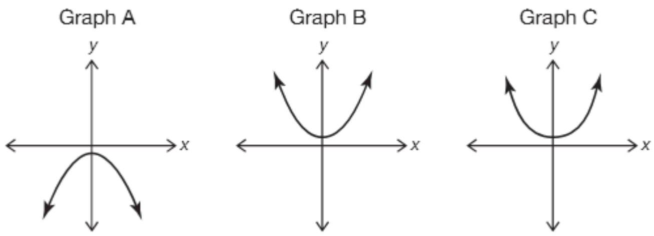
   a. Which graph(s) could be the graph of $f(x) = 2x^2$?



   b. Which graph(s) could be the graph of $f(x) = -x^3 - x^2 + 6x$?



   c. Which graph(s) could be the graph of $f(x) = x^4 + 1$?



Be prepared to share your solutions and methods.